Introduction to Mathematics and Modeling

lecture 8

The cross product
This week

1. Section 12.4: the cross product
2. Section 12.5: lines and planes in space
The cross product – introduction

Section 12.4

1.1

Definition

Let \( u = (u_1, u_2, u_3) \) and \( v = (v_1, v_2, v_3) \) be two vectors in \( \mathbb{R}^3 \). The cross product van \( u \) and \( v \) is defined as

\[
\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).
\]

- The Dutch name for the cross product is \textit{uitproduct} or \textit{uitwendig product}.
- The cross product can be computed using this trick:

\[
\begin{bmatrix}
  u_1 & u_2 & u_3 \\
  v_1 & v_2 & v_3
\end{bmatrix}
\]
For all $u, v, w \in \mathbb{R}^n$ and $r, s \in \mathbb{R}$ we have

1. $(ru) \times (sv) = (rs)(u \times v)$
2. $u \times (v + w) = u \times v + u \times w$
3. $u \times v = -(v \times u)$
4. $(v + w) \times u = v \times u + w \times u$
5. $0 \times u = u \times 0 = 0$
6. $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

- Property 4 can be proved with properties 2 and 3.
Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors. If $\theta$ is the acute positive angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin \theta.$$ 

- Acute means: $\theta \leq \pi$, hence $\sin \theta \geq 0$. 
The cross product – geometry

Theorem

For all vectors \( \mathbf{u} \) and \( \mathbf{v} \) we have \( \mathbf{u} \times \mathbf{v} \perp \mathbf{u} \) and \( \mathbf{u} \times \mathbf{v} \perp \mathbf{v} \).

- Vector \( \mathbf{u} \times \mathbf{v} \) is perpendicular to the plane through \( \mathbf{u} \) and \( \mathbf{v} \).
- The length of \( \mathbf{u} \times \mathbf{v} \) is \( |\mathbf{u}| \cdot |\mathbf{v}| \sin \theta \).
- The right-hand rule determines the direction of \( \mathbf{u} \times \mathbf{v} \).
The area of a parallelogram

**Theorem**

Let \( \mathbf{u} \in \mathbb{R}^3 \) and \( \mathbf{v} \in \mathbb{R}^3 \) be the edges of a parallelogram \( P \). Then the area of \( P \) is equal to \( |\mathbf{u} \times \mathbf{v}| \).

- Observe that \( \sin \theta = \frac{h}{|\mathbf{v}|} \), so \( h = |\mathbf{v}| \sin \theta \).
- The area of \( P \) is
  \[
  |\mathbf{u}| \cdot h = |\mathbf{u}| \cdot |\mathbf{v}| \sin \theta = |\mathbf{u} \times \mathbf{v}|.
  \]
Example

Find the area of the triangle $D$ with vertices $P = (1, -1, 0)$, $Q = (2, 1, -1)$ and $R = (-1, 1, 2)$. 
Problem

Let $S$ be a point in space and let $\ell$ be a line through $P$ with direction vector $v$. Find the distance $d$ of $S$ to $\ell$.

Method 1: Use the projection of $u = \overrightarrow{PS}$ on $\ell$:

Works in $\mathbb{R}^n$ for every $n$

$$d = |\mathbf{h}| = \left| \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \right|$$

Method 2: Use the cross product:

Only works in $\mathbb{R}^3$

$$d = |\mathbf{u}| \sin \theta = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{v}|}.$$
Example

Find the distance of $S = (1, 1, 5)$ to the line

\[ \ell : \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t. \]
Definition

A parametrisation of the plane $\mathcal{M}$ is a function of the form

$$\mathbf{p} + s\mathbf{v} + t\mathbf{w}, \quad s, t \in \mathbb{R}$$

- The vector $\mathbf{p}$ is called a **support vector** and the vectors $\mathbf{v}$ and $\mathbf{w}$ are called **direction vectors**.
Example 7

Find a parametrisation of the plane through the points $A = (0, 0, 1)$, $B = (2, 0, 0)$ and $C = (0, 3, 0)$. 
Problem

Find an equation of a plane $M$ given by a parametrisation

$$p + sv + tw,$$

where $P$ is a point of $M$ and $p = \overrightarrow{OP}$.

**Method 1:** Three-point method: observe that $P$, $Q = p + v$ and $R = p + w$ are three points of $M$. This gives three equations involving $x$, $y$, $z$, $s$ and $t$. Eliminate $s$ and $t$ to find one equation in $x$, $y$ and $z$.

**Method 2:** Compute a normal vector $\mathbf{n} = \mathbf{v} \times \mathbf{w}$ of $M$, then

$$M: \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0.$$
Example

Find an equation of the plane through the points \( A = (0, 0, 1), \)
\( B = (2, 0, 0) \) and \( C = (0, 3, 0) \).
**Theorem**

*Two different non-parallel planes intersect in a line.*

- Non-parallel means: the normals of both planes have different directions.
- If the planes are called $M$ and $N$, then the intersection line is denoted as follows:
  \[ \ell = M \cap N. \]
- A line in space can be regarded as the intersection line of two planes, in other words: it is the solution of a system of two equations:
  \[ \ell: \begin{cases} \ ax + by + cz = d, \\ px + qy + rz = s. \end{cases} \]
Intersection line of two planes

Cross product method:

- The normal vectors $\mathbf{n}_1$ of $M_1$ and $\mathbf{n}_2$ of $M_2$ are perpendicular to the intersection line, so the cross product of $\mathbf{n}_1$ and $\mathbf{n}_2$ is a direction vector of the intersection line.
- Any point that satisfies both equations is a support vector of the intersection line.